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FREE BOUNDARY PROBLEMS AND VARIATIONAL INEQUALITIES.(U)
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FREE BOUNDARY PROBLEMS AND VARIATIONAL INEQUALITIES

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LEVEL II

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ABSTRACT

The study of steady fluid flow through porous media leads to the consideration of elliptic free boundary problems in the unknown function "piezometric head". An integration along the vertical direction transforms these free boundary problems into variational (or quasi-variational) inequalities, which are easier to study both from the theoretical and from the numerical point of view.

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SIGNIFICANCE AND EXPLANATION

A typical free boundary problem involves motion of a fluid in a region, the boundary of the fluid being unknown and having to be determined as part of the solution. Problems of this type involving flow of fluids in porous media have important applications in ground water management, oil reservoir technology, and seepage of water through dams. Apart from certain special cases that can be solved analytically, computational methods for solving such problems have usually involved a laborious trial and error procedure - guess the position of the free boundary, and improve the guess by iteration.

In 1971, Baiocchi showed that many of these problems could be reformulated as variational inequalities. The use of variational inequalities (variational problems where the solution is subject to inequality constraints) and their generalization, quasi-variational inequalities, has led to profound developments, such as existence and uniqueness proofs for several classical free boundary problems, as well as for many new problems, and also very effective numerical methods. The advantage of the method from a computational point of view is that the position of the free boundary appears automatically in the calculation, without the need for guessing and iterating. These developments are surveyed in this report.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

FREE BOUNDARY PROBLEMS AND VARIATIONAL INEQUALITIES

C. Baiocchi*

Part 1. The simplest type of free boundary problem.

Let us consider the following problem:

PROBLEM A. Given:

(1.1) D: a bounded smooth domain in $\mathbb{R}^n$,

(1.2) f, g, ψ: smooth functions on $\bar{D}$; $\psi < g$ on ∂D.

We look for a couple $\{\Omega, u\}$ such that:

(1.3) Ω is an open smooth subset of D,

(1.4) u is a smooth function on $\bar{\Omega}$,

(1.5) $-\Delta u = f$ in Ω,

(1.6) $u = g$ on $\partial\Omega \cap \partial D$,

(1.7) $u = \psi$ and $\nabla u = \nabla\psi$ on $\partial\Omega \cap D$.

Problem A is the simplest f.b.p. (free boundary problem): on a domain Ω whose boundary is partially unknown we must solve a problem with too many conditions (see (1.7)) on the unknown part of $\partial\Omega$.

Remark 1.1. Condition (1.7) is equivalent to:

(1.7') $u = \psi$ and $\frac{\partial u}{\partial \nu} = \frac{\partial \psi}{\partial \nu}$ on $\partial\Omega \cap D$ ($\frac{\partial}{\partial \nu}$ denotes the outward normal derivative).

Remark 1.2. A very similar problem is the one where, instead of just one function g, we prescribe a couple $\{g_1, g_2\}$ and a partition of ∂D, $\{\partial_1 D, \partial_2 D\}$; then we replace (1.6) by:

(1.6') $u = g_1$ on $\partial_1 D \cap \partial\Omega$; $\frac{\partial u}{\partial \nu} = \frac{\partial g_2}{\partial \nu}$ on $\partial_2 D \cap \Omega$.

Remark 1.3. Under suitable assumptions on the data, the maximum principle will imply that:

(1.8) $u > \psi$ in Ω.

If we denote by $\tilde{u}$ the function defined by:

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$$(1.9) \qquad \tilde{u}(x) = \begin{cases} u(x) & \text{for } x \in \overline{\Omega} , \\ \psi(x) & \text{for } x \in \overline{D} \setminus \overline{\Omega} , \end{cases}$$

we can reformulate problem A in the following form:

PROBLEM B. Given D, f, g, ψ as in (1.1), (1.2), we look for $\tilde{u}$ such that

$$(1.10) \qquad \tilde{u} \text{ is smooth in } \overline{D} \; ; \; \tilde{u} \geq \psi \text{ in } \overline{D} ,$$

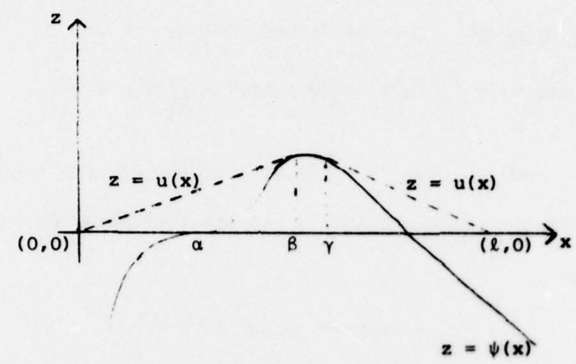
$$(1.11) \qquad \tilde{u} = g \text{ on } \partial D ,$$

$$(1.12) \qquad \begin{cases} \text{Setting } \Omega = \{x \in D | \; u(x) > \psi(x)\} \text{ we} \\ \text{have } -\Delta u = f \text{ in } \Omega . \end{cases}$$

As a physical situation which leads to mathematical problems of this type let us consider an elastic membrane above a region $D \subset R^2$ which is stretched along ∂D at the height $z = g(x,y)$, subjected to a system of vertical forces $F = f(x,y)$ and forced to stay above an obstacle $z = \psi(x,y)$; let C be the "contact region" and Ω be the complement of C in D, say $\Omega = D \setminus C$; then, assuming that the membrane is homogeneous and with unitary elastic coefficient, the shape $z = u(x,y)$ of the membrane at the equilibrium position must satisfy problem A (or problem B).

Remark 1.4. Problem A, as well as problem B, does not contain all physical conditions which must be imposed; we will come back to that point in a moment.

Let us consider the analogous one-dimensional problem with the simplest assumptions on the data; say the problem of an elastic string of length ℓ, fixed at its endpoints, stretched upon an obstacle $z = \psi(x)$ (the string is supposed weightless and no forces act on it); see Figure 1.1 for which $D =]0,\ell[$ and $\Omega =]0,\beta[\cup]\gamma,\ell[$.



<u>Figure 1.1</u>: *An elastic string stretched over an obstacle.*

In the picture is drawn the true physical solution, which can be obtained geometrically by drawing the straight lines from (0,0) and $(\ell,0)$ tangent to $z = \psi(x)$; however it is easy to see that, if ψ meets horizontally the x-axis at the point α, another (non physical) solution of problem B is given by $u_*(x) = \begin{cases} 0 & \text{for } 0 \le x \le \alpha , \\ \psi(x) & \text{for } \alpha \le x \le \gamma , \\ u(x) & \text{for } \gamma \le x \le \ell , \end{cases}$

the set Ω being now $]0,\alpha[\, \cup \,]\gamma,\ell[$.

In order to see what type of physical condition we neglected in problem A, we have essentially three ways, which lead to three equivalent (and well posed!) mathematical problems(').

<u>1st way</u>: <u>Minimum energy principle</u>. Let K be the set of all "admissible" configurations of "finite elastic energy" in the Sobolev space $H^1(D)$:

(1.13) $$K = \{v(x,y) \in H^1(D);\ v = g \text{ on } \partial D;\ v \ge \psi \text{ in } D\} ,$$

and let $J(v)$ be the energy associated with v:

(1.14) $$J(v) = \frac{1}{2} \iint_D |\nabla v|^2 dxdy - \iint_D f\, v\, dxdy ;$$

then the equilibrium shape $\tilde{u}$ must satisfy:

(1.15) $$\tilde{u} \in K ;\ J(\tilde{u}) \le J(v) \quad \forall\, v \in K ,$$

and (1.15) is a well posed problem (minimum of a quadratic coercive functional on a non empty closed convex set).

(') we come back to the general two-dimensional problem

2nd way: Virtual work principle. Let us define, for any u,v:

$$(1.16) \qquad a(u,v) = \iint_D \nabla u \cdot \nabla v \, dxdy \; ; \; L(v) = \iint_D f \, v \, dxdy \quad .$$

Remark that $a(u, u-v)$ and $-L(u-v)$ are the work of the elastic forces (resp. of the forces f) in order to pass from the configuration u to the configuration v; then the equilibrium shape $\tilde{u}$ must satisfy:

$$(1.17) \qquad \tilde{u} \in K; \quad a(\tilde{u}, \tilde{u}-v) \leq L(\tilde{u}-v) \quad \forall\, v \in K \quad .$$

(1.17) is a variational inequality, which has a unique solution (STAMPACCHIA's theorem, see [38]; in our present case the result is quite obvious because of the symmetry of a, namely $a(u,v) = a(v,u) \; \forall\, u,v$; we can remark that (1.17) is the EULER inequality of (1.15), so that problems (1.15) and (1.17) are equivalent.

3rd way: Balance of forces and reactions. Let us denote by $\mu(x,y)$ the reaction of the obstacle; we must have $\mu = -\Delta\tilde{u} - f$, so that:

$$(1.18) \qquad -\Delta\tilde{u} \geq f \quad \text{in} \quad D \quad ,$$

(because μ must be upward directed) and

$$(1.19) \qquad \text{support}(-\Delta\tilde{u} - f) \subset \{(x,y) \in D \,|\, \tilde{u}(x,y) = \psi(x,y)\} \; ,$$

(because we can have reaction just in the contact region); to (1.18), (1.19) we must obviously add $u \in K$. The problem in this formulation is often written:

$$(1.20) \qquad \tilde{u} \in H^1(D) \; ; \; \tilde{u} = g \quad \text{on} \quad \partial D \quad ;$$

$$(1.21) \qquad \tilde{u} \geq \psi \; ; \; -\Delta\tilde{u} \geq f \; ; \; (\tilde{u}-\psi)(-\Delta\tilde{u} - f) = 0 \quad \text{in} \quad D \quad ;$$

but this is a heuristic formula, the product $(\tilde{u}-\psi)(-\Delta\tilde{u} - f)$ having no meaning in general; on the contrary $\tilde{u} \in K$ and (1.18), (1.19) is a well posed problem (it is very easy to show that it is equivalent both to (1.15) and to (1.17)).

Remark 1.5. From (1.18), (1.19) we get obviously (1.12); the reverse implication is in general false, as we showed by the one-dimensional counterexample; in particular it was the strengthened form (1.18), (1.19) of (1.12) which was needed (compare with Remark 1.4).

In order to see that the (unique) solution of the variational inequality (1.17) solves problem B we still need a "regularity" result: in (1.10) we did not specify how smooth $\tilde{u}$

must be, but problem A suggests that along $\partial\Omega \cap D$ $\tilde{u}$ and ψ and their gradients must agree; now, from $u \geq \psi$ and the definition of Ω (see (1.12)) it is clear that, if ψ is a C^1 function, this agreement will hold if for instance we have:

(1.22) The solution $\tilde{u}$ of (1.17) is C^1 .

Actually, a well known result of LEWY and STAMPACCHIA (see [32]) implies that, under reasonable hypotheses of smoothness in (1.1), (1.2), the solution $\tilde{u}$ of (1.17) has second derivatives in $L^q(D)$ for any $q < +\infty$; in particular (1.22) holds and (1.21) has an obvious meaning.

Before discussing a more difficult f.b.p. let us sum up the results of this section: The LEWY-STAMPACCHIA regularity result is the key for interpreting the solution of a variational inequality as the solution of a f.b.p.; this was already pointed out in [32], where the f.b.p. solved by a variational inequality was a problem of lubrication (instead of the problem for the elastic membrane.)

<u>Historical and bibliographical note</u>. Variational inequalities like (1.17), when we assume that K is a closed convex non empty subset of a Hilbert space H, $a(u,v) = (u,v)_H$ (i.e., the scalar product in H) and $L(v) = (f,v)_H$ with $f \in H$ fixed, give rise to the problem of the projection of f on H; and the generalizations for L a linear continuous functional on H, and $a(u,v)$ bilinear continuous coercive and <u>symmetric</u> on $H \times H$ are quite obvious. The generalization to the case of <u>non symmetric $a(u,v)$</u> (which does not correspond to a minimum problem) is due to STAMPACCHIA [38] (see also LIONS-STAMPACCHIA [33] for parabolic inequalities). The possiblity of interpreting the solution of a variational inequality as the solution of a f.b.p. was not obvious, and in fact it is strictly connected to the regularity results; this idea was firstly developed by LEWY-STAMPACCHIA [32], and since then a lot of papers have been dedicated to the argument; see e.g., [16], [34].

Concerning the problem of regularity for free boundaries, (remark that we did not discuss the smoothness required in (1.3)) in recent years great progress has been made; see e.g., [18], [29], [30].

For further details on variational inequalities one can see the books [6] (recently appeared) and [31] (which will appear soon).

Part 2. A f.b.p. in hydraulics.

Let us consider the problem of a liquid flowing through a porous dam under the following schematic assumptions: The dam is homogeneous, isotropic, incompressible, with vertical plane parallel walls, on a horizontal impervious base; the liquid is *incompressible*; the flow is irrotational and steady (in particular the heights of the reservoirs in the picture are constant).

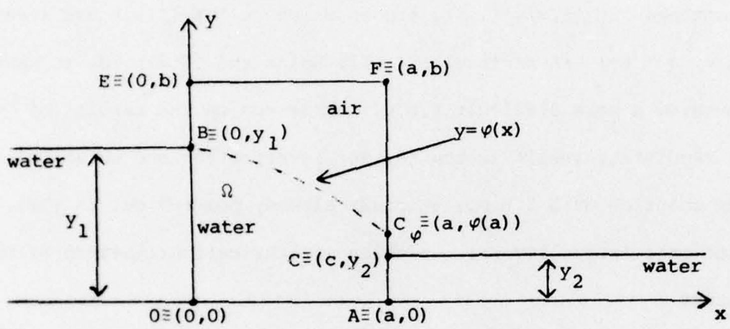


Figure 2.1: Porous flow through a rectangular dam.

Figure 2.1 shows the cross section of the dam;

$$D =]0,a[\times]0,b[\quad \text{is the dam,}$$

$$\Omega = \{(x,y) \in D \mid y < \varphi(x)\} \quad \text{is the flow region} \quad .$$

We remark that $\varphi(x)$ (and in particular the height of the point C_φ are unknown!).

Let us denote by $p(x,y)$ the pressure at the (x,y) point of D (the atmospheric pressure being measured by zero); DARCY's law implies that (assuming unitary some physical coefficients) if we set:

(2.1) $$u(x,y) = p(x,y) + y \quad \text{in } \overline{\Omega} \quad ,$$

we have

(2.2) $$\overrightarrow{\text{velocity}} = -\nabla u \quad \text{in } \Omega \quad ,$$

(we denote by Ω the interior of the wetted region) so that we get for u the following relations:

(2.3) $$\Delta u = 0 \quad \text{in} \quad \Omega \ ,$$

(because of (2.2) and the incompressibility);

(2.4) $$\frac{\partial u}{\partial \nu} = 0 \quad \text{along} \quad \overline{OA} \quad \text{and along} \quad \{y = \varphi(x)\}$$

(because they are streamlines),

(2.5) $$u = y_1 \quad \text{along} \quad \overline{OB}; \ u = y_2 \quad \text{along} \quad \overline{AC} \ ,$$

(we assume hydrostatic pressure on $\overline{OB}$, $\overline{AC}$);

(2.6) $$u = y \quad \text{along} \quad \{y = \varphi(x)\} \quad \text{and along} \quad \overline{CC_\varphi} \ ,$$

(the pressure is atmospheric).

It is obviously a f.b.p.: along the line $y = \varphi(x)$ (which is the unknown part of $\partial\Omega$) we impose both Dirichlet and Neumann conditions (compare with (1.7') and Remark 1.1; see also Remark 1.2 about the condition on $\overline{OA}$); the unique (but very important) difference with respect to Problem A is in the fact that the Neumann condition on the free boundary is $\frac{\partial u}{\partial \nu} = 0$ instead of $\frac{\partial u}{\partial \nu} = \frac{\partial y}{\partial \nu}$ (remark that now we must choose $\psi(x,y) \equiv y$, in order to have $u = \psi$ on $\partial\Omega \cap D$)(').

Remark 2.1. By assuming a little regularity on u (e.g., $u \in C_o(\overline{\Omega})$) we easily get from the maximum principle:

(2.7) $$u(x,y) > y \quad \text{in} \quad \Omega \ ,$$

(compare with (1.8)); on the other hand (2.7) means that the pressure must be strictly positive in Ω, which is physically obvious.

Let us reformulate the problem in terms of the pressure p (recall that p is defined on the whole of $\overline{D}$, and vanishes outside $\overline{\Omega}$); we will assume the continuity of p (which is physically obvious) and the fact (also obvious from the physical point of view) that the flow has a finite kinetic energy (see (2.9) below).

Problem C. Given a, b, y_1, y_2 with:

(') Remark (with notations of Remark 1.2) that we could choose $\partial_1 D = \overline{OA}$; $\partial_2 D = \partial D \setminus OA$; $g_1(x,y) = \max(y_1,y)$ on $\overline{OB}$; $g_1(x,y) = b$ on $\overline{EF}$; $g_1(x,y) = \max(y_2,y)$ on $\overline{AF}$; and $g_2 \equiv 0$. We have $\psi \le g$ on $\partial_1 D$ (instead of $\psi < g$) but this does not give additional problems.

(2.8) $$a > 0;\ 0 \le y_2 < y_1 \le b \quad ,$$

we look for a function p such that, setting $D =]0,a[\times]0,b[$, and denoting the spaces of continuous (continuously differentiable) functions on D vanishing on ∂D by $C_o(\overline{D})$ $(C_o^1(\overline{D}))$,

(2.9) $$p \in C_o(\overline{D}) \cap H^1(D) \quad ,$$

(2.10) $$p(0,y) = (y_1-y)^+;\ p(a,y) = (y_2-y)^+;\ p(b,x) = 0 \quad ,$$

(2.11) $$p(x,y) \ge 0 \quad \text{in } \overline{D} \quad ,$$

(2.12) $$p(x,y) > 0,\ y_* \in]0,y[\quad \text{imply} \quad p(x,y_*) > 0 \quad (')$$

and such that, setting:

(2.13) $$\Omega = \{(x,y) \in D \mid p(x,y) > 0\} \quad ,$$

we have:

(2.14) $$\begin{cases} \iint_\Omega \nabla(p+y)\cdot\nabla v \, dxdy = 0 \quad , \\ \forall\, v \in C_o^1(]0,a[\times \mathbb{R}) \quad . \end{cases}$$

Remark 2.2. (2.14) is the usual weak formulation for:

(2.15) $$\Delta(p+y) = 0 \quad \text{in } \Omega;\ \frac{\partial(p+y)}{\partial\nu} = 0 \quad \text{on } \partial\Omega \cap (D \cup \overline{OA}) \quad .$$

Remark 2.3. The meaning of (2.12) is that the set Ω defined through (2.13) (which is open thanks to (2.9)) must have the form:

(2.16) $$\Omega = \{(x,y) \in D \mid y < \varphi(x)\} \quad ,$$

for a suitable choice of φ; remark however that (2.12) does not impose any regularity property for that φ (φ must a priori be just a function lower-semi-continuous because of (2.9)).

We have already pointed out that the double condition on the free boundary ($p = 0$ and $\frac{\partial(p+y)}{\partial\nu} = 0$) is different from the one we encountered in problem A and B; now let us point

(') This hypothesis is a "regularity assumption" on the set Ω defined through (2.13) above; see also Remark 2.3.

out that, if we ask for the treatment of Problem C under the variational inequality form, the LEWY-STAMPACCHIA Theorem gives a negative answer: if the function p were the solution of a variational inequality, we would have $\frac{\partial p}{\partial \nu} = 0$ (instead of $\frac{\partial (p+y)}{\partial \nu} = 0$) along $\{y = \varphi(x)\}$. An other way of seeing this is the fact that p cannot have (as required from the LEWY-STAMPACCHIA Theorem) second derivatives in $L^q(D)$; in fact, from (2.14), choosing $v \in C_o^\infty(D)$, we get:

$$(2.17) \qquad \begin{cases} -\Delta p = D_y \chi_\Omega, & \text{in the distribution sense on } D , \\ \text{where } \chi_\Omega & \text{is the characteristic function of } \Omega \text{ in } D . \end{cases}$$

Let us give the (very simple) proof of (2.17). Equation (2.14) with $v \in C_o^\infty(D)$ gives (we denote by $\langle \ , \ \rangle$ the pairing between distributions and test functions on D; recall that $p \equiv 0$ outside Ω):

$$0 = \iint_\Omega \nabla(p+y)\cdot\nabla v \, dxdy = \iint_\Omega \nabla p\cdot\nabla v \, dxdy + \iint_\Omega \nabla y\cdot\nabla v \, dxdy =$$

$$= \iint_D \nabla p\cdot\nabla v \, dxdy + \iint_D \chi_\Omega v_y \, dxdy = \langle -\Delta p, v\rangle + \langle \chi_\Omega, v_y\rangle =$$

$$= \langle -\Delta p, v\rangle - \langle D_y\chi_\Omega, v\rangle = \langle -\Delta p - D_y\chi_\Omega, v\rangle .$$

Formula (2.17) suggests taking an integral of p with respect to y as a new unknown function, in order to destroy the "bad" operator D_y acting on χ_Ω; so let us define:

$$(2.18) \qquad w(x,y) = \int_y^b p(x,t)\,dt , \quad (x,y) \in \overline{D} .$$

Easy computations give:

$$(2.19) \qquad \Delta w = \chi_\Omega \text{ in } D ,$$

$$w(x,b) \equiv 0; \; w(0,y) = \int_y^b (y_1-t)^+ dt; \; w(a,y) = \int_y^b (y_2-t)^+ dt ,$$

$$w_{yy} = -1 \text{ on } \overline{AB} \Rightarrow w_{xx} = \Delta w - w_{yy} = 0 \text{ on } \overline{AB} ,$$

so that, if we denote by $g(x,y)$ any (smooth) function such that $g(x,b) \equiv 0$,

$g(0,y) = \int_y^b (y_1 - t)^+ dt$, $g(a,y) = \int_y^b (y_2 - t)^+ dt$, $g(x,0)$ linear (so that $g(x,0) = \frac{y_2^2}{2} + \frac{y_1^2 - y_2^2}{2a}(a-x)$) we get

$$w = g \text{ on } \partial D . \tag{2.20}$$

Now, recalling (2.12), (2.13) and $p \geq 0$, we get:

$$w \geq 0 \text{ on } \overline{D} , \tag{2.21}$$

$$\Omega = \{(x,y) \in D \mid w(x,y) > 0\} , \tag{2.22}$$

so that we can replace (2.19) by:

$$-\Delta w \geq -1; \; w(-\Delta w - 1) = 0 \text{ on } D . \tag{2.23}$$

Now (2.21), (2.22) look like (1.21) with $\psi \equiv 0$, $f \equiv -1$; because of the regularity of g (whose first derivatives are Lipschitz continuous on each edge of D) we can conclude:

Theorem 2.1. There exists a unique $w \in C^1(\overline{D})$, with second derivatives in $L^q(D)$ $\forall q < +\infty$, such that (2.20), (2.21), (2.23) hold.

This theorem implies that Problem C has at most one solution; if such a solution exists, it can be evaluated starting from w and setting Ω defined through (2.22) and p defined through (compare with (2.18));

$$p(x,y) = -(D_y w)(x,y) \quad \forall (x,y) \in \overline{D} ; \tag{2.24}$$

the main difficulty is the fact that it is not obvious that Ω defined through (2.22) has the form (2.16). Actually we can prove that Ω defined through (2.22) has the form (2.16) and moreover the corresponding φ is very smooth; precisely:

$$\begin{cases} \varphi \in C_0([0,a]); \; \varphi \text{ is strictly decreasing; } \varphi \text{ is} \\ \text{analytic in }]0,a[; \; \varphi(0) = y_1; \; \varphi'(0) = 0; \; \varphi(a) > y_2, \; \varphi'(a) = -\infty \end{cases} \tag{2.25}$$

so that we can conclude:

Theorem 2.2. Problem C has a unique solution; the upper boundary of Ω has the further regularity properties listed in (2.25).

Remark 2.4. If we ask for further regularity properties of p (which a priori satisfies just (2.9)) we find:

$$p_x \in L^q(D) \quad \forall q < +\infty; \; p_y \in L^\infty(D) \quad , \tag{2.26}$$

and such regularity is "optimal" because actually we have:

$$p_x \notin L^\infty(D); \; p_y \notin C_0(\overline{D}) \quad . \tag{2.27}$$

Now let us point out a typical computational difficulty which always appears when a f.b.p. is solved by means of variational inequalities. Many papers and books deal with the numerical solution of variational inequalities (e.g., see [27]); so that we can assume that we have constructed a family $\{w_h(x,y)\}_{h>0}$ of functions which, as $h \to 0$, converges in some sense, to the solution $w(x,y)$ of (2.20), (2.21), (2.23); setting e.g., $p_h(x,y) = -D_y w_h(x,y)$ (compare with (2.24)) we will also have $p_h \to p$ in some sense; but how can we approximate Ω (which is the true unknown of our problem)? The naive idea of defining $\Omega_h = \{(x,y) \in D \mid w_h(x,y) > 0\}$ does not work, no matter in what topology $w_h \to w$: an obvious counter-example is given by choosing $w_h(x,y) = w(x,y) + h$ (so that $w_h \to w$ in the C^∞ topology!) so that $\Omega_h \equiv D$. However we can bypass this difficulty by means of the following theorem.

Theorem 2.3. Let $w_h \to w$ uniformly; and let ε_h be such that

$$\varepsilon_h \searrow 0 \quad , \quad \frac{\|w_h - w\|_{L^\infty(D)}}{\varepsilon_h} \to 0 \quad . \tag{2.28}$$

Setting $\Omega_h = \{(x,y) \in D \mid w_h(x,y) > \varepsilon_h\}$ we have "$\Omega_h \to \Omega$ from the interior", say $\Omega = \lim_{h \to 0} \Omega_h$ (lim in the set theoretic sense) and $\Omega_h \subset \Omega$ for h sufficiently small.

Historical and bibliographical note. Problems like Problem B (perhaps in a less precise form) are very ancient; apart from some analog solutions (HELE-SHAW model) they were solved numerically, before the advent of computers, by means of the so called "inverse method" or "hodographic method"; see e.g., the book [36]. For general problems in hydraulics see e.g., the books [13], [28], [36]. After the advent of computers a more efficient numerical method has been proposed; the idea is the following one: we fix an initial guess, $y = \varphi_0(x)$, for the

free boundary; in the corresponding set Ω_0 we solve the problem (2.14) and the two first relations of (2.10); the solution $p_0(x,y)$ will not satisfy the condition $p_0(x, \varphi_0(x)) = 0$ (otherwise we have already solved our problem!) but we can use the values of $p_0(x, \varphi_0(x))$ in order to modify $\varphi_0(x)$ and get a new guess $\varphi_1(x)$ (e.g., we could define $\varphi_1(x) = \varphi_0(x) + p_0(x, \varphi_0(x)))$; a fixed point for the transformation $\varphi_0 \to \varphi_1$ will give the solution of our problem. This is just a heuristic approach (e.g., with the formula $\varphi_1(x) = \varphi_0(x) + p_0(x, \varphi_0(x))$ we will never modify the starting value for $\varphi(a)$!) but numerically it works (for references to this method, and others, see [21]).

The transform (2.18) was suggested in [2]; beside the existence and uniqueness result for the problem it suggested a new numerical treatment (see [20], [8]) which is theoretically correct and which competes very well with the previous heuristic methods both in simplicity of programming and in speed of execution.

The problem of the convergence $\Omega_h \to \Omega$ was firstly solved, in a particular case, in [11]; the general result of Th. 2.3 has been exploited in [12], in connection with a parabolic f.b.p. (diffusion-absorption of oxygen in tissues). Remark that (2.28) requires the knowledge of (or an estimate for) the L^∞ norm for the error; such an estimate (the best one: $\|w - w_h\|_{L^\infty} = \sigma(h^{2-\varepsilon})\ \forall\varepsilon > 0$, h mesh-size, for piece-wise linear elements) has been obtained independently in [3] and [35]. Some improvements of Theorem 2.3 (say L^σ or H^1 norms instead of L^∞ norm) can be given (H. Brézis, personal talk, unpublished). Concerning (2.25), (2.26), (2.27) see [18], [29], [30]. For other related results see part 3 as well as [6] and its bibliography.

Part 3. Recent progresses and Bibliography.

Because of space restriction I will give a few details just for some other problems of elliptic type related to hydraulics; however let me point out that transforms like (2.18) have been adapted to many different problems; e.g., for fluids dynamics problems (flow past a symmetric aerofoil; see [15], [16]), for heat-conduction problems (like the classical Stefan problems; see [22]), for non-steady hydraulics problems, which differ from both elliptic problems and parabolic ones (in fact the problem in the interior involves only space-derivatives, the evolution condition being just on the free boundary) see [41], and, for more details and references, see the talk of Professor FRIEDMAN [24]).

For a general overview on the class of f.b.p. which, by means of a suitable change of unknown functions (like (2.18)), can be translated into variational inequalities see [5] and the Appendix 4, Vol. II of [6].

Now let me give some more details concerning steady problems in hydraulics:

A) Variable permeability can be taken into account with some restrictions; e.g., horizontal layers as well as vertical layers can be studied by this method; see [8], [9], [14].

B) Phenomena like evaporation, or partially pervious bases are studied in [23], [37].

C) Capillarity offers some theoretical difficulties; the corresponding numerical treatment seems to suggest that the theory must be correct. See [19].

D) More general geometry can be taken into account; when the right wall is vertical we get (instead of one variational inequality) a family of variational inequalities depending on a real parameter (the discharge of the dam); the value for the parameter is found by imposing a "regularity condition" (see [8], [9]). When also the right wall is sloping the problem becomes a quasi-variational inequality, i.e., a family of variational inequalities depending on a functional parameter (see [4]).

E) Three dimensional problems, in the case of vertical walls, do not offer new difficulties; see [39]; for non vertical walls the corresponding quasi-variational inequality has been studied in [26].

F) A different way of treating the problem in n-dimensions and with very general geometry has been proposed in [1]; the method is based on the search for the minimum of kinetic energy

in a suitable class of supersolutions; the fact that the point of minimum satisfies the problem is proved by a "balayage" which uses locally the transform (2.18).

G) Another approach, which leads to a nonlinear equation (instead of a variational inequality) has been proposed in [42].

H) Problems of fluid flow through a channel are studied in [40].

I) Problems for coastal aquifers can also be studied; in [8], [9] was treated the problem with vertical cliffs; [7], [10] treat the problem with general geometry and in the presence of wells sunk into the aquifer and pumping water; conditions which ensure that just fresh water is pumped, and not the salt water which infiltrates under the aquifer, are also discussed.

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The study of steady fluid flow through porous media leads to the consideration of elliptic free boundary problems in the unknown function "piezometric head". An integration along the vertical direction transforms these free boundary problems into variational (or quasi-variational) inequalities, which are easier to study both from the theoretical and from the numerical point of view.

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